Angular momentum

We have already seen momentum, and we could loosely describe it as the tendency for moving objects to keep moving. More precisely, an object's momentum is not changed unless there is a net force acting on the object. If we take a system which has nothing pushing or pulling on it from the outside, the momentum is also conserved. In addition, we can have a system which does feel external pushes and pulls, but if those pushes and pulls cancel out momentum is also conserved.

We can introduce an analogous concept of angular momentum, which can loosely be described as the tendency for objects to keep rotating unless something from outside the system pushes or pulls on it. For a system with no external pushes or pulls acting on it, the angular momentum is conserved.

There is an important difference, though: if there are external pushes or pulls that cancel out momentum is conserved – this is not necessarily true of angular momentum. The condition that external "rotational" pushes or pulls cancel out is a more advanced concept which we will deal with in DL #8. In DL #7, you can simply assume that the angular momentum of our systems are conserved.

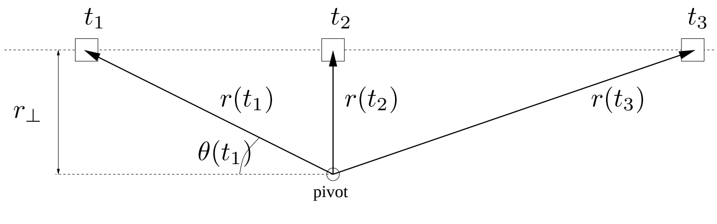
Momentum versus angular momentum

To define angular momentum, we need to have a centre of rotation. Then the angular momentum of an object will tell us about the tendency of an object to rotate about *this particular point.* By choosing different points, we can make the angular momentum different.

Any object that has ordinary momentum also has an angular momentum associated with it. To find the magnitude of the angular momentum L we need to know the momentum (i.e. both magnitude and direction) and the displacement between the object and the pivot. We call this displacement the "lever arm" and denote it by $\boldsymbol{r}$. To get the magnitude of the angular momentum, we multiply the magnitude of the momentum p by the component of $\boldsymbol{r}$ perpendicular to the momentum, $r_\perp$.

$$L = pr_\perp = mvr_\perp$$

To see how this works, it helps to consider an example. Consider an object travelling at a constant velocity as shown below. The object is drawn at three different times: t_1, t_2 and t_3, with t_1 being the earliest and t_3 the latest. The lever arm $\boldsymbol{r}$ is shown at each of these times, and is continually changing. The component of $\boldsymbol{r}$ perpendicular to $\boldsymbol{p}$ is also shown as $r_\perp$, and it remains the same.



It may seem odd that something travelling in a *straight* line has *angular* momentum. But this object is rotating around the pivot: if you look at the angle θ it is getting bigger clockwise as the object is travelling. Choosing the zero for θ to be to the left was arbitrary, but regardless of where you choose zero to be θ will be increasing in the clockwise direction in this example.

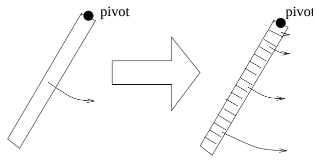
Now we have found the magnitude, but we still need to associate a direction with angular momentum. To do this, we use the same righthand rule that we used to give $\boldsymbol{\omega}$ a direction. So in the previous example, the object was "rotating" clockwise. By using RHR #1 we see that the direction for angular momentum is *into* the page.

Test your understanding: What would the angular momentum of the object be if the pivot was on the line instead? Try and figure this out yourself; if you get stuck ask at office hours.

Rotational inertia

While we can put the pivot anywhere we want, the reason we call it the pivot is because we usually choose it so that the entire system is rotating around it. If modelling a door, I would put the pivot somewhere on the hinge, as the hinge does not move and the rest of the door rotates around it. If I was looking at a bike wheel rotating, I would put the pivot at the centre as the rest of the bike wheel rotates around it. Other choices are not *wrong*, they are just (usually) less *convenient*.

So let us look at a situation like a rod rotating about its end. Let us model the rod by breaking it up into lots of tiny little masses stuck together, like so:



We know that the magnitude of angular momentum for any little bit is $L = mvr_\perp$. But because each little mass is travelling in a circular path, $r_\perp$ is the distance down the rod. But how did I know it was travelling in a circular path? Well, the little masses cannot be getting any closer together or further apart as the rod is a rigid object. Because the angular momentum for each of the little blocks points in the same direction, we can add them together:

$$L = m_1 r_{\perp,1} v_1 + \ldots + m_n r_{\perp,N} v_N$$

Because all these objects are moving in a circle about the pivot and the pieces of the rod cannot move relative to one another that they all have the same angular velocity $\boldsymbol{\omega}$. Going back to our previous reading, we see that

$$v_i = \omega\, r_{\perp,i}$$

Putting this back into our calculation of angular momentum gives

$$\begin{aligned} L &= m_1(\omega r_{\perp,1}) r_{\perp,1} + \ldots + m_N(\omega r_{\perp,N}) r_{\perp,N} \\ &= \left(m_1 r_{\perp,1}^2 + \ldots + m_N r_{\perp,N}^2\right) \omega \end{aligned}$$

The term inside the bracket depends on the mass of the blocks and where they are. Without imagining the rod, the term in the brackets depends on how big the mass is *and* how the mass is distributed away from the axis of rotation. The further away a mass is, the bigger the value of the bracket is as the distance comes in $r_\perp^2$. By changing how far away the masses are from the axis of rotation, we change the term in the bracket.

The bracketed term is important enough to have its own name: rotational inertia. We denote it by I. Now our formula for L looks a lot simpler:

$$L = I\omega.$$

Because we use the same RHR to assign a direction to $\boldsymbol{\omega}$ and $\boldsymbol{L}$, we can make this a *vector* equation instead:

$$\boldsymbol{L} = I\boldsymbol{\omega}.$$

Notice the similarity to $\boldsymbol{p} = m\boldsymbol{v}$. I plays the role of telling you how "hard" it is to change the rotational velocity $\boldsymbol{\omega}$ in the same way that m tells you how hard it is to change the "normal" velocity $\boldsymbol{v}$. The big difference between these two is that an objects mass cannot change suddenly (unless it becomes two or more different objects, by exploding), whereas by moving mass around you can change I. In particular, because m cannot change a constant $\boldsymbol{p}$ implies a constant $\boldsymbol{v}$. But because I can change, constant $\boldsymbol{L}$ does *not* imply constant $\boldsymbol{\omega}$.

The thing that makes rotational inertia really useful is that people have tabulated what the rotational inertia is for different shapes already. You have such a table on page 38 of your course notes. One key point to remember is that for a point mass, $I = Mr_\perp^2$.