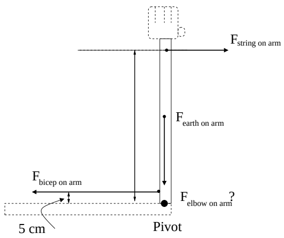
FNT #6 – Angular momentum and torque

April 27, 2005

Question 1 – the bicep problem

a) and b)

Let us start by drawing our force body diagram:



My artistic abilities are not quite up to those of the people that write the physics FNTs, but hopefully you see that we are only considering the forearm, the upper arm and hand are not important (which is why they are dotted). Notice that one force does not appear on the diagram – the force of the elbow on the arm – as I don't know which way it goes yet. So this is not a final answer.

Which forces generate a torque about the pivot point shown (the big dot)? Well, we know that the force of the earth on the arm is parallel to the lever arm so it does not generate a torque. We don't know which way the force of the elbow points, but we do know that its lever arm is zero. So only two forces generate a torque:

- The force of the string on the arm (into the page)
- The force of the biceps on the arm (out of the page)

Since the arm is not rotating, and these forces are perpendicular to their lever arms I know

$$F_{\text{string on arm}} r_{\text{string to pivot}} = F_{\text{bicep on arm}} r_{\text{bicep to pivot}}$$

or that

$$F_{\text{bicep on arm}} = \frac{r_{\text{string to pivot}}}{r_{\text{bicep to pivot}}} F_{\text{string on arm}}$$

Placing numbers into this equation yields

$$F_{mboxbiceponarm} = \frac{30\,\text{cm}}{5\,\text{cm}}\,(180\,\text{N}) = 6 \times 180\,\text{N} = 1080\,\text{N}$$

Now I know the magnitude of each of the other three forces, and I know the sum of the forces has to be zero as the arm is not accelerating. While I *could* do this on an extended force body diagram, I find it easier to use an non-extended force body diagram. We can get the force by components:

Horizontal components

$$\begin{aligned} \sum \vec{F}_x &= 0 \\ \Rightarrow \vec{F}_{\text{bicep on a},x} + \vec{F}_{\text{string on a},x} + \vec{F}_{\text{earth on a},x} + \vec{F}_{\text{elbow on a},x} &= 0 \\ (-1080\,\text{N}) + (180\,\text{N}) + 0\,\text{N} + \vec{F}_{\text{elbow on a},x} &= 0 \\ \vec{F}_{\text{elbow on a},x} &= 900\,\text{N} \end{aligned}$$

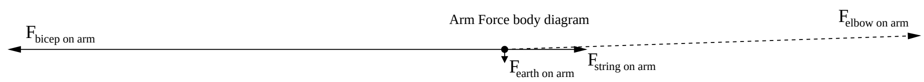
i.e. The elbow acts on the arm with 900 Newtons to the *left*.

Vertical components

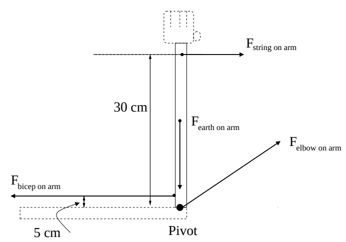
$$\begin{aligned} \sum \vec{F}_y &= 0 \\ \Rightarrow \vec{F}_{\text{bicep on a},y} + \vec{F}_{\text{strng on a},y} + \vec{F}_{\text{earth on a},y} + \vec{F}_{\text{elbow on a},y} &= 0 \\ (0\,\text{N}) + (0\,\text{N}) + (-30\,\text{N}) + \vec{F}_{\text{elbow on a},y} &= 0 \\ \vec{F}_{\text{elbow on a},y} &= 30\,\text{N} \end{aligned}$$

i.e. The elbow supplies 30 Newtons of force in an *upward* direction to balance out gravity.

Let us draw a force diagram, just so we can see where the different forces go:



I should emphasise this force diagram is to *scale*. This is incredibly hard to draw, because if I want to see my vertical components at all, my horizontal components have to be huge! Because it is so hard, I am just going to give a vague (i.e. not to scale) extended force diagram:



Notice that although it is not to scale, I have been careful to make sure that the horizontal and the vertical forces balance (people look for this on the quiz).

Summary

This was a hard problem. One of the things that makes it harder is that the way it is asked you would think that you could do part a) and then part b). This is incorrect, the only way of getting to the answer is to do both together (i.e. balance forces and torques) and then draw the force diagrams. We will try and fix this for future quarters.

Also notice that the direction on the arm from the elbow is surprising. The elbow force is a special type of force known as a *constraint force*, as it is constraining the arm to stick to the elbow. These forces arise when you have an object that is attached and will supply a large amount of force to stop the object breaking them. Other examples of constraint forces are the force of floor on you (which stops you falling), force of table on book (which stops the book accelerating through the table), force of bat on ball (which stops the ball travelling straight through the bat). Almost all constraint forces arise from an object trying to stay in one piece while resisting another object trying to go through it. These constraint forces will point in any direction, sometimes with very large magnitudes, to prevent the object from moving.

That is why I did not start with the "constraint force". I know that I have one long range force that is relevant:

- The force of the Earth on the arm.

The only things in direct contact with the arm are then

- The string from the spring (force of string on arm)
- The biceps (force of biceps on arm)
- The elbow (force of elbow on arm)[1]

The string and the bicep have obvious directions (the same direction as the bicep or string is pointing). The elbow force is a constraint force, and will point anyway necessary to stop the arm breaking through it, and that is why I left it to last.

Some other general points about this sort of problem are

- Although all the information you need is on an extended force body diagram, it is easy to forget forces or get forget to sum the forces. It is a good idea to do a separate (non-extended) force body diagram
- To find the net torques an extended force body diagram is essential.
- Even a large force at a small lever arm has a small torque.

[1] Some of the physiology students may wish to point out that the arm and elbow are now directly touching, and that a cartilage lies between them. All that changes is that this would be the force of cartilage on arm instead.

c)

Explain in complete English sentences why the bone of the upper arm (or elbow) exerts such a large force on the lower arm

Only two forces give a torque about the elbow, the string on the arm and the bicep on the arm. As the bicep has a much smaller lever arm, the value of the force must be much greater than the force of the string on the arm so that the torques balance. The elbow must balance the forces, and as the force from the bicep is much larger than the force from the string (all on the arm), the elbow must exert a large force to oppose the bicep.

(In summary, the reason is that the bicep has such a small "lever arm").

d) and e)

The *advantage* to having your bicep connected so low is that you allow your arm to bend more without your biceps getting in the way.

The *disadvantage* is that to lift things in your hand, you need very strong biceps!

Question 2 – the merry-go-round

Preliminary notes

*** To be written ***

a) Rotational inertia of student and merry-go-round

The rotational inertias simply sum:

$$I_{\text{total}} = I_{\text{student}} + I_{\text{mgr}}$$

The student is just a point mass (almost) at a distance $r = 3$ metres away. So we have

$$I_{\text{student}} = m_{\text{student}} r^2 = (70\,\text{kg})(3\,\text{m})^2 = 630\,\text{kg m}^2.$$

The merry-go-round is a uniform disk, so we have to look up in our course notes what the rotational inertia of a disk of mass m_{mgr} and radius R_{mgr} is:

$$I_{\text{mgr}} = \frac{1}{2} m_{\text{mgr}} R_{\text{mgr}}^2 = \frac{1}{2}(30\,\text{kg})(3\,\text{m})^2 = 135\,\text{kg m}^2$$

The rotational inertia of both together is then

$$I_{\text{total}} = 630\,\text{kg m}^2 + 135\,\text{kg m}^2 = 765\,\text{kg m}^2$$

b) Rotational speed of merry-go-round and student *after* student gets on

The initial angular momentum of the student is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

which is a fancy way of writing that the angular momentum of the student is the perpendicular component of the lever arm times the *linear momentum* (see the preliminary section). Doing the calculation yields:

$$\begin{aligned} \vec{p}_{\text{student}} &= m_{\text{student}} \vec{v}_{\text{student}} = (70\,\text{kg})(4\,\text{m/s}) = 280\,\text{kg m/s} \\ |\vec{L}_{\text{student}}| &= (280\,\text{kg m/s})(3\,\text{m}) = 840\,\text{kg m}^2/\text{s} \end{aligned}$$

This, together with the fact that this is a closed *angular momentum* system, gives us enough to start our angular momentum chart:

	L_i	$\Delta\vec{L}$	$\vec{L}_f$
Student	840 kg m^2/s ↑	?	?
MGR	0	?	?
Total	840 kg m^2/s ↑	0	840 kg m^2/s ↑

We know that because the student and the merry-go-round are going together, that the final angular velocities will be the same. i.e.

$$\begin{aligned} \vec{L}_{\text{final, total}} &= \vec{L}_{\text{final, student}} + \vec{L}_{\text{final, mgr}} \\ &= I_{\text{student}} \vec{\omega}_{\text{f}} + I_{\text{mgr}} \vec{\omega}_{\text{f}} \\ &= (I_{\text{student}} + I_{\text{mgr}}) \vec{\omega}_{\text{f}} \\ &= I_{\text{total}} \vec{\omega}_f \end{aligned}$$

But we calculated I_{total} in part a, and we know $\vec{L}_f = \vec{L}_i$! Hence we can solve for the final angular velocity:

$$\vec{\omega} = \frac{840\,\text{kg m}^2/\text{s (up)}}{765\,\text{kg m}^2} = 1.1\,\text{rad/s (up)}$$

At the end, we use the right hand rule to convert back to a direction of rotation (counter-clockwise as seen from above).

Note two things:

- I only needed conservation of angular momentum – I did not need the entire chart!

- This problem is almost exactly equivalent to the asteroid problem. Here the person and the merry-go-round were stuck together, so they had the same *angular velocity*, just like the two asteroids stuck together had the same linear velocity.

c) Stopping the merry-go-round

If it took 0.04s to come to a stop on the now spinning merry-go-round, what was the average torque applied to the merry-go-round in that time period?

I wrote the question out because we have to be careful about two things:

- The person has *not* come to a stop with respect to the ground. They are spinning around with the merry-go-round with the same angular velocity.
- We are looking for the torque applied to the merry-go-round, not to the system (the total torque applied to the system is zero!).

The final angular momentum of the merry-go-round is

$$\vec{L}_{\text{final,mgr}} = I_{\text{mgr}}\vec{\omega}_{\text{f}} = (135\,\text{kg m}^2)(1.01\,\text{rad/s (up)}) = 148.5\,\text{kg m}^2\text{/s (up)}$$

So the change in $\Delta\vec{L}$ is

$$\Delta\vec{L}_{\text{mgr}} = \vec{L}_{\text{final,mgr}} - \vec{L}_{\text{i,mgr}} = 148.5\,\text{kg m}^2\text{/s (up)}$$

We also know

$$\begin{aligned} \Delta\vec{L} &= \left(\sum\tau\right)_{\text{ave on mgr}} \Delta t \\ \Rightarrow \left(\sum\tau\right)_{\text{ave on mgr}} &= \frac{\Delta\vec{L}}{\Delta t} = \frac{148.5\,\text{kg m}^2\text{/s (up)}}{0.04s} \\ &= 3713\,\text{kg m}^2\text{/s}^2\ \text{(up)} \end{aligned}$$

Let us place the values we found into our angular momentum chart, just so we can see where they all go:

	L_i	$\Delta\vec{L}$	$\vec{L}_f$
Student	840 kg m^2/s ↑	?	?
MGR	0	148.5 kg m^2/s ↑	148.5 kg m^2/s ↑
Total	840 kg m^2/s ↑	0	840 kg m^2/s ↑

Of course we could easily fill in the other two cells, in a number of ways. I just thought that this would help people that did not like algebra see where the numbers above came from. It is also a good point to emphasise that only the initial and the final states matter: the average torque and the time taken to speed the merry-go-round up did not appear in this equation!

This problem is analogous to the car crash problem, and looking at the effect of Δt there.

d)

The initial kinetic energy was just the translational kinetic energy of the student:

$$\text{K.E.}_{\text{before}} = \frac{1}{2} m_{\text{student}} v_i^2 = \frac{1}{2}(70\,\text{kg})(4\,\text{m/s})^2 = 560\,\text{J}$$

Afterwards, there is the rotational energy of the entire system (nothing is travelling along, just spinning around):

$$\text{K.E.}_{\text{after}} = \frac{1}{2} I_{\text{total}} \omega_f^2 = \frac{1}{2}(765\,\text{kg m}^2)(1.1\,\text{rad/s})^2 = 463\,\text{J}$$

We find the change by

$$\Delta\text{K.E.} = \text{K.E.}_{\text{after}} - \text{K.E.}_{\text{before}} = -97\,\text{J}$$

This is an *inelastic* collision, as energy has been lost to other systems (such as noise, heat, sound, etc).